

# On Hall subgroups of a finite group

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March 3, 2013

## Abstract

In the paper new criteria of existence and conjugacy of Hall subgroups of finite groups are given.

## 1 Introduction

Throughout this paper, all groups are finite and  $G$  always denotes a finite group. We use  $\mathbb{P}$  to denote the set of all primes,  $\pi$  is a non-empty subset of  $\mathbb{P}$  and  $\pi' = \mathbb{P} \setminus \pi$ . A natural number  $n$  is called a  $\pi$ -number if  $(n, p) = 1$  for any prime  $p \in \pi'$ . If  $|G|$  is a  $\pi$ -number, then  $G$  is said to be a  $\pi$ -group.

Let  $A$  and  $B$  be subgroups of  $G$ . Then  $A$  is said to permute with  $B$  if  $AB = BA$ . If  $A$  permutes with all (Sylow) subgroups of  $B$ , then  $A$  is called *quasinormal* (*S-quasinormal*, respectively) in  $B$ .

A group  $G$  is said to be:

- (a) an  $E_\pi$ -group if  $G$  has a Hall  $\pi$ -subgroup;
- (b) a  $C_\pi$ -group if  $G$  is an  $E_\pi$ -group and any two Hall  $\pi$ -subgroups of  $G$  are conjugate;

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\*Research of the first author is supported by a NNSF grant of China (Grant # 11071229) and Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences.

<sup>†</sup>Research of the second author supported by Chinese Academy of Sciences Visiting Professorship for Senior International Scientists (grant No. 2010T2J12)

Keywords: finite group, soluble group, Sylow subgroup, Hall subgroup, quasinormal subgroup,  $E_\pi$ -group,  $C_\pi$ -group,  $D_\pi$ -group.

Mathematics Subject Classification (2010): 20D20

(c) a  $D_\pi$ -group if  $G$  is an  $E_\pi$ -group and any  $\pi$ -subgroup of  $G$  is contained in some Hall  $\pi$ -subgroup of  $G$ .

A group  $G$  is said to be  $\pi$ -separable if  $G$  has a chief series

$$1 = G_0 \leq H_1 \leq \dots \leq H_{t-1} \leq H_t = G, \quad (*)$$

where each index  $|H_i : H_{i-1}|$  is either a  $\pi$ -number or a  $\pi'$ -number.

The most important result of the theory of  $\pi$ -separable groups is the following classical result.

**Theorem A** (P. Hall, S.A. Čunihin). *Any  $\pi$ -separable group is a  $D_\pi$ -group.*

Our main goal here is to prove the following generalization of this theorem.

**Theorem 1.1.** *Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

where  $|H_{i+1} : H_i|$  is either a  $\pi$ -number or a  $\pi'$ -number for all  $i = 1, \dots, t$ . Suppose also that  $G$  has a subgroup  $T$  such that  $H_1 T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is quasinormal in  $T$ , for all  $i = 1, \dots, t$ , then  $G$  is a  $D_\pi$ -group.

The following theorem is one of the main steps in the proof of Theorem 1.1.

**Theorem 1.2.** *Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

where  $|H_{i+1} : H_i|$  is either a  $\pi$ -number or a  $\pi'$ -number for all  $i = 1, \dots, t$ . Suppose also that  $G$  has a subgroup  $T$  such that  $H_1 T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is  $S$ -quasinormal in  $T$ , for all  $i = 1, \dots, t$ , then  $G$  is a  $C_\pi$ -group.

Theorem 1.1 strengthens the main result in [1]. Note also that Example 1.1 in [1] shows that, under the conditions in Theorems 1.1 or 1.2 the group  $G$  is not necessary  $\pi$ -separable.

## 2 Preliminaries

**Lemma 2.1.** *Let  $N$  be a normal  $C_\pi$ -subgroup of  $G$ .*

- (i) *If  $G/N$  is a  $C_\pi$ -group, then  $G$  is a  $C_\pi$ -group (S. A. Čunihin [3]).*
- (ii) *If  $G/N$  is an  $E_\pi$ -group, then  $G$  is an  $E_\pi$ -group*
- (iii) *If  $G$  has a nilpotent Hall  $\pi$ -subgroup, then  $G$  is a  $D_\pi$ -group ([19]).*
- (iv) *If  $G$  has a Hall  $\pi$ -subgroup with cyclic Sylow subgroups, then  $G$  is a  $D_\pi$ -group (S. A. Rusakov [20]).*

**Lemma 2.2.** *Let  $N$  be a normal  $C_\pi$ -subgroup of  $G$  and  $N_\pi$  a Hall  $\pi$ -subgroup of  $N$ .*

(i) If  $G$  is a  $C_\pi$ -group, then  $G/N$  is a  $C_\pi$ -group (See [4, Lemma 9]).

(ii) If every Sylow subgroup of  $N_\pi$  is cyclic and  $G/N$  is a  $D_\pi$ -group, then  $G$  is a  $D_\pi$ -group (See [5] or [6, Chapter IV, Theorem 18.17]).

(iii)  $G$  is a  $D_\pi$ -group if and only if  $G/N$  is a  $D_\pi$ -group (See [7]).

**Lemma 2.3** (O. Kegel [8]). *Let  $A$  and  $B$  be subgroups of  $G$  such that  $G \neq AB$  and  $AB^x = B^x A$ , for all  $x \in G$ . Then  $G$  has a proper normal subgroup  $N$  such that either  $A \leq N$  or  $B \leq N$ .*

Let  $A$  be a subgroup of  $G$ . A subgroup  $T$  is said to be a minimal supplement of  $A$  in  $G$  if  $AT = G$  but  $AT_0 \neq G$  for all proper subgroups  $T_0$  of  $G$ .

The following lemma is obvious.

**Lemma 2.4.** *If  $N$  is normal in  $G$  and  $T$  is a minimal supplement of  $N$  in  $G$ , then  $N \cap T \leq \Phi(T)$ .*

**Lemma 2.5** (P. Hall [10]). *Suppose that  $G$  has a Hall  $p'$ -subgroup for each prime  $p$  dividing  $|G|$ . Then  $G$  is soluble.*

Let  $A$  and  $B$  be subgroups of  $G$  and  $\emptyset \neq X \subseteq G$ . Following [9], we say that  $A$  is  $X$ -permutable (or  $A$   $X$ -permutes) with  $B$  if  $AB^x = B^x A$  for some  $x \in X$ .

The following lemma is also evident.

**Lemma 2.6.** *Let  $A, B, X$  be subgroups of  $G$  and  $K \trianglelefteq G$ . If  $A$  is  $X$ -permutable with  $B$ , then  $AK/K$  is  $XK/K$ -permutable with  $BK/K$  in  $G/K$ .*

**Lemma 2.7** (Kegel [11]). *If a subgroup  $A$  of  $G$  permutes with all Sylow subgroups of  $G$ , then  $A$  is subnormal in  $G$ .*

**Lemma 2.8** (H. Wielandt [12]). *If a  $\pi$ -subgroup  $A$  of  $G$  is subnormal in  $G$ , then  $A \leq O_\pi(G)$ .*

**Lemma 2.9** (V. N. Knyagina and V. S. Monakhov [17]). *Let  $H, K$  and  $N$  be subgroups of  $G$ . If  $H$  is a Hall subgroup of  $G$ , then*

$$N \cap HK = (N \cap H)(N \cap K).$$

**Lemma 2.10.** *Let  $U \leq B \leq G$  and  $G = AB$ , where the subgroup  $A$  permutes with  $U^b$  for all  $b \in B$ . Then  $A$  permutes with  $U^x$  for all  $x \in G$ .*

**Proof.** Since  $G = AB$ ,  $x = ab$  for some  $a \in A$  and  $b \in B$ . Hence

$$AU^x = AU^{ab} = Aa(U^b)a^{-1} = a(U^b)a^{-1}A = U^x A.$$

**Lemma 2.11** [13, Chapter I, Lemma 1.1.19]. *Let  $A, B \leq G$  and  $G = AB$ . Then  $G_p = A_p B_p$  for some  $G_p \in \text{Syl}_p(G)$ ,  $A_p \in \text{Syl}_p(A)$  and  $B_p \in \text{Syl}_p(B)$ .*

**Lemma 2.12.** *Let  $X$  be a normal  $C_\pi$ -subgroup of  $G$ . Suppose that  $G$  has a subgroup series*

$$1 = H_0 \leq H_1 \leq \dots \leq H_{t-1} \leq H_t = G$$

and a supplement  $T$  of  $H_1$  in  $G$  such that  $H_i$   $X$ -permutes with every Sylow subgroup of  $T$  for all  $i = 1, \dots, t$ . If each index  $|H_{i+1} : H_i|$  is either a  $\pi$ -number or a  $\pi'$ -number, then  $G$  is an  $E_\pi$ -group.

**Proof.** Consider the subgroup series

$$1 = H_0X/X \leq H_1X/X \leq \dots \leq H_{t-1}X/X \leq H_tX/X = G/X \quad (1)$$

in  $G/X$ . If  $T \leq X$ , then  $G = H_1T = XH_1$ . Hence  $G$  is an  $E_\pi$ -group by Lemma 2.1(2). Now let  $T \not\leq X$  and  $P/X$  be a Sylow  $p$ -subgroup of  $TX/X$ . Then by Lemma 2.11 there are Sylow  $p$ -subgroups  $T_p$  of  $T$  and  $X_p$  of  $X$  such that  $T_pX_p$  is a Sylow  $p$ -subgroup of  $TX$  and  $T_pX_pX = T_pX = P$ . Hence by Lemma 2.6,  $H_iX/X$  permutes with every Sylow subgroup of  $TX/X$  for all  $i = 1, \dots, t$ . On the other hand, since

$$|H_{i+1}X/X : H_iX/X| = |H_{i+1}X : H_iX| = |H_{i+1} : H_i| \cdot |X \cap H_{i+1} : X \cap H_i|,$$

every index of Series (1) is either  $\pi$ -number or  $\pi'$ -number. Hence the assertion follows from Theorem B in [1].

**Lemma 2.13** (see Lemma in [14, Chapter A]). *Let  $H$ ,  $K$  and  $N$  be subgroups of  $G$ . If  $HK = KH$  and  $HN = NH$ , then  $H\langle K, N \rangle = \langle K, N \rangle H$ .*

### 3 Base proposition

**Proposition 3.1.** *Let  $X$  be a normal  $C_\pi$ -subgroup of  $G$  and  $A$  a subgroup of  $G$  such that  $|G : A|$  is a  $\pi$ -number. Suppose that  $A$  has a Hall  $\pi$ -subgroup  $A_0$  such that either  $A_0$  is nilpotent or every Sylow subgroup of  $A_0$  is cyclic. Suppose that  $A$   $X$ -permutes with every Sylow  $p$ -subgroup of  $G$  for all primes  $p \in \pi$  or for all primes  $p \in \pi \setminus \{q\}$  for some prime  $q$  dividing  $|G : A|$ . Then  $G$  is a  $C_\pi$ -group.*

**Proof.** Assume that this proposition is false and let  $G$  be a counterexample of minimal order. Then  $|\pi \cap \pi(G)| > 1$ .

(1)  $G/R$  is a  $C_\pi$ -group for any non-identity normal subgroup  $R$  of  $G$ .

In order to prove this assertion, in view of the choice of  $G$ , it is enough to show that the hypothesis is still true for  $(G/R, AR/R, XR/R)$ . First note that  $|G/R : AR/R| = |G : AR|$  is a  $\pi$ -number, and  $A_0R/R$  is a Hall  $\pi$ -subgroup of  $AR/R$  since

$$|AR/R : A_0R/R| = |AR : A_0R| = |A : A \cap A_0R| = |A : A_0(A \cap R)|.$$

On the other hand,  $XR/R \simeq X/X \cap R$  is a  $C_\pi$ -group by Lemma 2.2 (i), and either  $A_0R/R \simeq A_0/R \cap A_0$  is nilpotent or every Sylow subgroup of  $A_0R/R$  is cyclic. Finally, let  $P/R$  be a Sylow  $p$ -subgroup of  $G/R$ , where  $p \in \pi \setminus q$ . Then for some Sylow  $p$ -subgroup  $G_p$  we have  $G_pR/R = P/R$ . Hence  $AR/R$   $X$ -permutes with  $P/R$  by Lemma 2.6. Therefore the hypothesis holds for  $(G/R, AR/R, XR/R)$ .

(2)  $X = 1$ .

Indeed, if  $X \neq 1$ , then  $G/X$  is a  $C_\pi$ -group by (1). Hence  $G$  is  $C_\pi$ -group by Lemma 2.1 (i), a contradiction.

(3)  $G$  has a proper non-identity normal subgroup  $N$ .

Let  $p \in \pi \cap \pi(G)$ , where  $p \neq q$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . First assume that  $AP = G$ . Since  $|\pi \cap \pi(G)| > 1$ , there is a prime  $r \in \pi \cap \pi(G)$  such that  $r \neq p$ , so  $r$  does not divide  $|G : A|$ . Let  $R$  be a Sylow  $r$ -subgroup of  $G$ . Then for any  $x \in G$  we have  $AR^x = R^x A$ . Hence  $R \leq A_G$ . Since  $G$  is not a  $C_\pi$ -group,  $A \neq G$  by Lemma 2.1 (iii)(iv). Hence  $1 \neq A_G \neq G$ . Now suppose that  $AP \neq G$ . By (2),  $P^x A = AP^x$  for all  $x \in G$ . Hence we have (3) by Lemma 2.3.

(4)  $N$  is a  $C_\pi$ -group.

In view of the choice of  $G$  it is enough to prove that the hypothesis holds for  $(N, A_1)$ , where  $A_1 = A \cap N$ . Since  $|AN : A| = |N : A \cap N|$ ,  $|N : A_1|$  is a  $\pi$ -number. On the other hand,  $A_0 \cap N$  is a Hall  $\pi$ -subgroup of  $N$  since

$$|A \cap N : A_0 \cap N| = |A_0(A \cap N) : A_0|.$$

It clear also that either  $A_0 \cap N$  is nilpotent or every Sylow subgroup of  $A_0 \cap N$  is cyclic. Now let  $N_r$  be any Sylow  $r$ -subgroup of  $N$ , where  $r \in \pi \setminus \{q\}$ . Then for some Sylow  $r$ -subgroup  $G_r$  of  $G$  we have  $N_r = G_r \cap N$  and

$$N \cap G_r = (A \cap N)(N \cap G_r) = A_1 N_r = N_r A_1$$

by Lemma 2.9. Therefore the hypothesis holds for  $(N, A_1)$ .

Finally, in view of (1) and (4),  $G$  is a  $C_\pi$ -group by Lemma 2.1 (i), which contradicts the choice of  $G$ .

**Corollary 3.2.** *Let  $X$  be a normal  $C_\pi$ -subgroup of  $G$  and  $A$  a subgroup of  $G$  such that  $|G : A|$  is a  $\pi$ -number. Suppose that  $A$  has a Hall  $\pi$ -subgroup  $A_0$  such that either  $A_0$  is nilpotent or every Sylow subgroup of  $A_0$  is cyclic. Suppose that  $G$  has a subgroup  $T$  such that  $|G : T|$  is a  $\pi'$ -number,  $G = AT$  and  $A$   $X$ -permutes with every Sylow  $p$ -subgroup of  $T$  for all primes  $p \in \pi(G) \setminus \pi$ ,  $p \neq q \in \pi$ . Then  $G$  is a  $C_{\pi'}$ -group.*

**Proof.** Let  $p \in \pi(G) \setminus \pi$ ,  $p \neq q$  and  $P$  be a Sylow  $p$ -subgroup of  $G$ . Since  $G = AT$  and  $|G : T|$  is a  $\pi'$ -number, every Sylow  $p$ -subgroup  $T_p$  of  $T$  is a Sylow  $p$ -subgroup of  $G$ . Hence for some  $x \in G$  we have  $T_p = P^x$ , so  $A$  permutes with  $P$  by Lemma 2.10. Therefore Corollary 3.2 follows from Proposition 3.1.

**Corollary 3.3** ( see Theorem 1.2 in [9]). *Let  $X$  be a normal nilpotent subgroup of  $G$  and  $A$  a Hall  $\pi$ -subgroup of  $G$ . Suppose  $G = AT$  and  $A$   $X$ -permutes with every subgroup of  $T$ . Then  $G$  is a  $C_{\pi'}$ -group.*

**Corollary 3.4** (See Theorem B in [16]). *Let  $A$  be a Hall subgroup of a group  $G$  and  $T$  a minimal supplement of  $A$  in  $G$ . Suppose that  $A$  permutes with all Sylow subgroups of  $T$  and with all maximal subgroups of any Sylow subgroup of  $T$ . Then  $G$  is a  $C_{\pi'}$ -group.*

**Corollary 3.5** (see Theorem A\* in [1]). *Let  $H$  be a Hall  $\pi$ -subgroup of  $G$ . Let  $G = HT$  for some subgroup  $T$  of  $G$ , and  $q$  a prime. If  $H$  permutes with every Sylow  $p$ -subgroup of  $T$  for all primes  $p \neq q$ , then  $T$  contains a complement of  $H$  in  $G$  and any two complements of  $H$  in  $G$  are conjugate.*

**Proof.** It is clear that  $T \cap H$  is a Hall  $\pi$ -subgroup of  $T$ . Moreover, if  $P$  is a Sylow subgroup of  $T$  and  $HP = PH$ , then  $HP \cap T = P(H \cap T) = (H \cap T)P$ . Hence by Proposition 3.1,  $T$  is a  $C_\pi$ -group. Now Corollary 3.5 follows from Proposition 3.1.

**Corollary 3.6** (see [18]). *Let  $A$  be a Hall  $\pi$ -subgroup of  $G$ ,  $G = AT$  and  $A$  permutes with every subgroup of  $T$ . Then  $G$  is an  $E_{\pi'}$ -group.*

## 4 Proof of Theorem 1.1

Theorem 1.2 is a special case (when  $X = 1$ ) of the following theorem.

**Theorem 4.1.** *Let  $X$  be a normal  $C_\pi$ -subgroup of  $G$ . Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

*where  $|H_{i+1} : H_i|$  is either a  $\pi$ -number or a  $\pi'$ -number for all  $i = 1, \dots, t$ . Suppose that  $G$  has a subgroup  $T$  such that  $H_1T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is  $X$ -permutable with each Sylow subgroup of  $T$ , for all  $i = 1, \dots, t$ , then  $G$  is a  $C_\pi$ -group.*

**Proof.** Suppose that this theorem is false and let  $G$  be a counterexample of minimal order. By Lemma 2.12,  $G$  has a Hall  $\pi$ -subgroup  $S$ . Hence some Hall  $\pi$ -subgroup  $S_1$  of  $G$  is not conjugated with  $S$ . Without loss of generality, we may assume that  $H_1 \neq 1$ . Since  $|G : T|$  is  $\pi'$ -number, every Sylow  $p$ -subgroup  $P$  of  $T$ , where  $p \in \pi$ , is also a Sylow  $p$ -subgroup of  $G$ . We proceed the proof via the following steps.

(1)  $G/N$  is a  $C_\pi$ -group for every non-trivial quotient  $G/N$  of  $G$ .

We consider the subgroup series

$$1 = H_0N/N \leq H_1N/N \leq \dots \leq H_{t-1}N/N \leq H_tN/N = G/N \quad (2)$$

in  $G/N$ . Then  $(H_1N/N)(TN/N) = G/N$  and  $|G/N : TN/N| = |G : TN|$  is a  $\pi'$ -number. Moreover, by Lemma 2.6,  $H_iN/N$  is  $XN/N$ -permutable with every Sylow subgroup of  $TN/N$  for all  $i = 1, \dots, t$ . On the other hand, since

$$|H_{i+1}N/N : H_iN/N| = |H_{i+1}N : H_iN| = |H_{i+1} : H_i| \cdot |N \cap H_{i+1} : N \cap H_i|,$$

every index of the series (2) is either  $\pi$ -number or  $\pi'$ -number. Moreover,  $XN/N \simeq X/(X \cap N)$  is a  $C_\pi$ -group by Lemma 2.2 (i). All these show that the hypothesis holds for  $G/N$ . Hence in the case, where  $N \neq 1$ ,  $G/N$  is a  $C_\pi$ -group by the choice of  $G$ .

(2)  $O_{\pi'}(G) = 1 = O_\pi(G)$ .

Suppose that  $D = O_{\pi'}(G) \neq 1$ . Then by (1), there is an element  $x \in G$  such that  $S_1^x D = SD$ . But by the Schur-Zassenhaus theorem,  $S_1^x$  and  $S$  are conjugate in  $SD$ , which implies that  $S_1$  and  $S$  are conjugate in  $G$ . This contradiction shows that  $O_{\pi'}(G) = 1$ . Analogously, one can prove that  $O_{\pi}(G) = 1$ .

(3)  $X = 1$  (This follows from (1), Lemma 2.1 (i) and the choice of  $G$ ).

(4)  $T \neq G$ .

Suppose that  $T = G$ . Then by hypothesis and (3),  $H_1$  permutes with all Sylow subgroups of  $G$ . It follows from Lemma 2.7 that  $H_1$  is subnormal in  $G$ . Since  $H_1$  is either a  $\pi$ -group or a  $\pi'$ -group,  $H_1 \leq O_{\pi}(G)$  or  $H_1 \leq O_{\pi'}(G)$  by Lemma 2.8. It follows from (2) that  $H_1 = 1$ , which contradicts to our assumption about  $H_1$ . Hence (4) holds.

(5) *The hypothesis holds for  $T$ .*

Consider the subgroup series

$$1 = H_0 \cap T \leq H_1 \cap T \leq \dots \leq H_{t-1} \cap T \leq H_t \cap T = T \quad (3)$$

of the group  $T$ . Since

$$H_{i+1} = H_i T \cap H_{i+1} = H_i (H_{i+1} \cap T)$$

we have

$$|H_{i+1} : H_i| = |H_{i+1} \cap T : H_i \cap T|,$$

for all  $i = 1, \dots, t-1$  and  $|H_1 \cap T : H_0 \cap T| = |H_1 \cap T|$  divides  $|H_1 : 1|$ , we see that every index of the series (3) is either  $\pi$ -number or  $\pi'$ -number. Now let  $E$  be a Sylow subgroup of  $T$ . By (3) and the hypothesis,  $H_i E = E H_i$ . Hence

$$H_i E \cap T = E (H_i \cap T) = (H_i \cap T) E.$$

This shows that the hypothesis holds for  $T$ .

(6)  $T$  is a  $C_{\pi}$ -group.

Since  $T \neq G$  by (4), and the hypothesis holds for  $T$  by (5), the minimal choice of  $G$  implies that (6) holds.

(7)  $T$  is a  $E_{\pi'}$ -subgroup.

This follows from (3), (5) and Lemma 2.12.

(8) *Let  $T_{\pi'}$  be a Hall  $\pi'$ -subgroup of  $T$  and  $D$  a normal subgroup of  $G$ . Then  $T_{\pi'} \neq 1$ , and if either  $H_1 \leq D$  or  $T_{\pi'} \leq D$ , then  $D = G$ .*

Suppose that  $T_{\pi'} = 1$ . Then  $H_1$  is a Hall  $\pi'$ -subgroup of  $G$ . Therefore  $G$  is a  $C_{\pi}$ -group by Corollary 3.2, a contradiction. Hence  $T_{\pi'} \neq 1$ .

We show that the hypothesis holds for  $D$ . Consider the subgroup series

$$1 = D_0 \leq D_1 \leq \dots \leq D_{t-1} \leq D_t = D,$$

where  $D_i = H_i \cap D$  for all  $i = 1, \dots, t$ . Let  $T_0 = D \cap T$ . First we show that  $D_1 T_0 = D$ . If  $H_1 \leq D$ , then

$$D = H_1(D \cap T) = H_1 T_0 = D_1 T_0.$$

Now suppose that  $T_{\pi'} \leq D$ . In view of (3),  $H_1$  permutes with  $T_{\pi'}$ . Since  $H_1 T = G$ ,  $T \neq G$  and  $|G : T|$  is a  $\pi'$ -number,  $H_1$  is a  $\pi'$ -group. Hence  $H_1 T_{\pi'}$  is a Hall  $\pi'$ -subgroup of  $G$ . Therefore

$$D = (D \cap H_1 T_{\pi'})(D \cap T_{\pi}) = T_{\pi'}(D \cap H_1)(D \cap T_{\pi}) = (D \cap H_1)(T_{\pi'}(D \cap T_{\pi})) = (D \cap H_1)(D \cap T) = D_1 T_0.$$

It is also clear that  $|D : T_0|$  is a  $\pi'$ -number. Now let  $P$  be a Sylow  $p$ -subgroup of  $T_0$ . Then for some Sylow  $p$ -subgroup  $T_p$  of  $T$  we have  $P = T_p \cap D$ . Hence in view of Lemma 2.9,

$$D \cap H_i T_p = (D \cap H_i)(D \cap T_p) = D_i P = P D_i.$$

Thus for any  $p \in \pi$ ,  $D_i$  is permutable with every Sylow  $p$ -subgroup of  $T_0$  for all  $i = 1, \dots, t$ . Finally, since the number

$$|D_i : D_{i-1}| = |(D \cap H_i)H_{i-1} : H_{i-1}|$$

divides  $|H_i : H_{i-1}|$ , each index  $|D_i : D_{i-1}|$  is either a  $\pi$ -number or a  $\pi'$ -number. Therefore the hypothesis holds for  $D$ . Suppose that  $D \neq G$ . Then  $D$  is a  $C_\pi$ -group by the choice of  $G$ . Since either  $1 \neq H_1 \leq D$  or  $1 \neq T_{\pi'} \leq D$ ,  $G/D$  is a  $C_\pi$ -group by (1). It follows from Lemma 2.1 (i) that  $G$  is a  $C_\pi$ -group, which contradicts the choice of  $G$ . Hence, (8) holds.

*Final contradiction.* Since  $G = H_1 T$  and  $H_1$  permutes with all Sylow subgroups of  $T$  by (3),

$$H_1(T_{\pi'})^x = (T_{\pi'})^x H_1$$

for all  $x \in G$  by Lemma 2.10. Therefore by Lemma 2.3, either  $H_1^G \neq G$  or  $(T_{\pi'})^G \neq G$ . But in view of (8) both these cases are impossible. The contradiction completes the proof of the result.

**Corollary 4.2** (see Theorem 5.1 in [1]). *Let  $X$  be a normal  $\pi$ -separable subgroup of  $G$ . Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

*where  $|H_{i+1} : H_i|$  is either a  $\pi$ -number or a  $\pi'$ -number for all  $i = 1, \dots, t$ . Suppose that  $G$  has a subgroup  $T$  such that  $H_1 T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is  $X$ -permutable with each subgroup of  $T$ , for all  $i = 1, \dots, t$ , then  $G$  is a  $C_\pi$ -group.*

Theorem 1.1 is a special case (when  $X = 1$ ) of the following theorem.

**Theorem 4.3.** *Let  $X$  be a normal  $E_\pi$ -subgroup of  $G$  and  $X_\pi$  a Hall  $\pi$ -subgroup of  $X$ . Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

*where  $|H_{i+1} : H_i|$  is either a  $\pi$ -number or a  $\pi'$ -number for all  $i = 1, \dots, t$ . Suppose that  $G$  has a subgroup  $T$  such that  $H_1 T = G$  and  $|G : T|$  is a  $\pi'$ -number.*



(i) Suppose that the Sylow subgroups of  $X_\pi$  are cyclic. If  $H_i$  is  $X$ -permutable with each cyclic subgroup  $H$  of  $T$  of prime power order, for all  $i = 1, \dots, t$ , then  $G$  is a  $D_\pi$ -group.

(ii) Suppose that  $X$  is a  $D_\pi$ -subgroup. If  $H_i$  is  $X$ -permutable with each cyclic subgroup  $H$  of  $T$  of prime power order, for all  $i = 1, \dots, t$ , then  $G$  is a  $D_\pi$ -group.

**Proof.** (i) Suppose that this assertion is false and let  $G$  be a counterexample of minimal order. In view of Theorem 4.1,  $G$  is a  $C_\pi$ -group. Hence there is a  $\pi$ -subgroup  $U$  of  $G$  such that for any Hall  $\pi$ -subgroup  $E$  of  $G$  we have  $U \not\leq E$ .

(1)  $G/N$  is a  $D_\pi$ -group for any non-identity normal subgroup  $N$  of  $G$ .

Consider the subgroup series

$$1 = H_0N/N \leq H_1N/N \leq \dots \leq H_{t-1}N/N \leq H_tN/N = G/N.$$

It is clear that  $(H_1N/N)(TN/N) = G/N$ ,  $|G/N : TN/N|$  is a  $\pi'$ -number and  $|H_{i+1}N/N : H_iN/N|$  is either a  $\pi$ -number or a  $\pi'$ -number for all  $i = 1, \dots, t$  (see (1) in the proof of Theorem 4.1). Now let  $H/N$  be a cyclic subgroup of  $TN/N$  of prime power order  $|H/N|$ . Then  $H = N(H \cap T)$ . Let  $W$  be a group of minimal order with the properties that  $W \leq H \cap T$  and  $NW = H$ . If  $N \cap W \not\leq \Phi(W)$ , then for some maximal subgroup  $S$  of  $W$  we have  $(N \cap W)S = W$ . Hence  $H = NW = N(N \cap W)S = NS$ , a contradiction. Hence  $N \cap W \leq \Phi(W)$ . Since  $W/W \cap N \simeq H/N$  is a cyclic group of prime power order, it follows that  $W$  is cyclic group of prime power order. Hence  $H_i$  is  $X$ -permutable with  $W$ . Thus  $H_iN/N$  is  $XN/N$ -permutable with  $WN/N = H/N$  by Lemma 2.6. Therefore the hypothesis holds for  $G/N$ . But since  $N \neq 1$ ,  $|G/N| < |G|$  and so  $G/N$  is a  $D_\pi$ -group by the choice of  $G$ .

(2)  $X = 1$ .

Suppose that  $X \neq 1$ . Then  $G/X$  is a  $D_\pi$ -group by (1). Hence  $G$  is a  $D_\pi$ -group by Lemma 2.2 (iii), a contradiction. Thus we have (2).

(3)  $H_1$  permutes with every subgroup of  $U$ .

Let  $Z$  be any cyclic subgroup of  $U$  of prime power order  $p^n$ . Then  $p \in \pi$  and  $Z \leq G_p$  for some Sylow  $p$ -subgroup  $G_p$  of  $G$ . Since  $|G : T|$  is a  $\pi'$ -number, there is an element  $x = ht$  such that  $(G_p)^x \leq T$ . Hence  $H_1Z = ZH_1$  by Lemma 2.10, which in view of Lemma 2.13 implies that  $H_1U = UH_1$ .

(4)  $T \neq G$  (see the proof of (4) in the proof of Theorem 4.1).

(5)  $T$  is a  $D_\pi$ -group.

The hypothesis holds for  $T$  (see (5) in the proof of Theorem 4.1), so in view of (4) the minimal choice of  $G$  implies that we have (5).

(6)  $V = H_1U$  is a  $C_\pi$ -group.

Indeed,  $V$  is a group by (3), and since  $T \neq G$ ,  $H_1$  is a Hall  $\pi'$ -subgroup of  $V$ . Therefore we have (6) by (2) and Corollary 3.2.

(7)  $|V : T \cap V|$  is a  $\pi'$ -number and  $T \cap V$  is a  $C_\pi$ -group.

First note that  $|G : T| = |V : T \cap V|$  is a  $\pi'$ -number. But  $H_1$  is a Hall  $\pi'$ -subgroup of  $V$ . Hence  $V = H_1(T \cap V)$ , which implies that

$$|V : H_1| = |T \cap V : H_1 \cap T \cap V|$$

is a  $\pi$ -number. Hence  $A = H_1 \cap T \cap V$  is a Hall  $\pi'$ -subgroup of  $T \cap V$ . Finally, if  $W$  is any  $\pi$ -subgroup of  $T \cap V$ , then  $H_1W = WH_1$  by (3). Therefore

$$AW = (H_1 \cap T \cap V)W = (H_1W \cap T \cap V) = WA.$$

Hence  $T \cap V$  is a  $C_\pi$ -group by Proposition 3.1.

*Final contradiction for (i).* In view of (7),  $|V : T \cap V|$  is a  $\pi'$ -number and  $T \cap V$  is a  $C_\pi$ -group. Hence in view of (6), there is an element  $x \in V$  such that  $U^x \leq T \cap V \leq T$ . But by (5),  $T$  is a  $D_\pi$ -group. Hence for some Hall  $\pi$ -subgroup  $T_\pi$  of  $T$  we have  $U \leq T_\pi$ , which is a contradiction since  $T_\pi$  is clearly a Hall  $\pi$ -subgroup of  $G$ .

(ii) See the proof of (i) and use Lemma 2.2 (iii).

## 5 Groups with soluble Hall $\pi$ -subgroups

A group  $G$  is said to be:

- (a) an  $E_\pi^S$ -group ( $E_\pi^N$ -group) if  $G$  has a soluble (a nilpotent, respectively) Hall  $\pi$ -subgroup;
- (b) a  $C_\pi^S$ -group ( $C_\pi^N$ -group) if  $G$  has a soluble (nilpotent, respectively) Hall  $\pi$ -subgroup and  $G$  is a  $C_\pi$ -group;
- (c) a  $D_\pi^S$ -group if  $G$  is a  $C_\pi^S$ -group and any  $\pi$ -subgroup of  $G$  is contained in some Hall  $\pi$ -subgroup of  $G$ .

Our next results are new criteria for a group to be an  $E_\pi^S$ -group.

**Lemma 5.1.** *Let  $N$  be a normal  $E_\pi$ -subgroup of  $G$  and  $N_\pi$  a Hall  $\pi$ -subgroup of  $N$ . Suppose that  $N_\pi$  is nilpotent. If  $G/N$  is a  $D_\pi^S$ -group, then  $G$  is a  $D_\pi^S$ -group (See [4] or [6, Chapter IV, Theorem 18.15]).*

**Theorem 5.2.** *Let  $X$  be a normal  $E_\pi^N$ -subgroup of  $G$ . Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

*where  $|H_{i+1} : H_i|$  is divisible by at most one prime in  $\pi$ , for all  $i = 1, \dots, t$ . Suppose that  $G$  has a subgroup  $T$  such that  $H_1T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is  $X$ -permutable with each Sylow subgroup of  $T$ , for all  $i = 1, \dots, t$ , then  $G$  is an  $E_\pi^S$ -group.*

**Proof.** Suppose that this theorem is false and let  $G$  be a counterexample of minimal order.

(1) Every non-trivial quotient  $G/N$  of  $G$  is an  $E_\pi^S$ -group (See (1) in the proof of Theorem 4.1).

(2)  $X = 1$ .

Suppose that  $X \neq 1$ . Then  $G/X$  of  $G$  is an  $E_\pi^S$ -group by (1). Let  $E/X$  be a soluble  $\pi$ -Hall subgroup of  $G/X$ . Then, by Lemma 5.1,  $E$  is a  $D_\pi^S$ -group and if  $U$  is a Hall subgroup of  $E$ , then  $U$  is a Hall subgroup of  $G$ . Hence  $G$  is an  $E_\pi^S$ -group, a contradiction. Thus we have (2).

(3)  $T \neq G$ .

Suppose that  $T = G$ . Then by hypothesis and (2),  $H_1$  permutes with all Sylow subgroups of  $G$ . Hence  $H_1$  is subnormal in  $G$  by Lemma 2.7, so

$$1 < H_1 \leq O_{\pi' \cup \{p\}}(G)$$

for some  $p \in \pi$  by Lemma 2.8. Therefore  $O_{\pi' \cup \{p\}}(G)$  is a non-identity normal  $D_\pi^N$ -group-subgroup of  $G$ , which as in the proof of (2), conducts us to the contradiction. Thus (3) holds.

*Final contradiction.* In view of (2), the hypothesis is true for  $T$  (see (5) in the proof of Theorem 4.1). Hence  $T$  is an  $E_\pi^S$ -group by (3) and the choice of  $G$ . But since  $|G : T|$  is a  $\pi'$ -number, any Hall  $\pi$ -subgroup of  $T$  is a Hall  $\pi$ -subgroup of  $G$  as well. Hence  $G$  is an  $E_\pi^S$ -group, a contradiction.

The theorem is proved.

**Theorem 5.3.** Let  $X$  be a normal  $E_\pi^N$ -subgroup of  $G$ . Suppose that  $G$  has a subgroup series

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

where  $|H_{i+1} : H_i|$  is divisible by at most one prime in  $\pi$ , for all  $i = 1, \dots, t$ . Suppose that  $G$  has a subgroup  $T$  such that  $H_1 T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is  $X$ -permutable with each cyclic subgroup  $H$  of  $T$  of prime power order, for all  $i = 1, \dots, t$ , then  $G$  is a  $D_\pi^S$ -group.

**Proof.** Suppose that this theorem is false and let  $G$  be a counterexample of minimal order. Then  $|\pi \cap \pi(G)| > 1$ . Without loss we may suppose that  $H_1 \neq 1$  and  $H_{t-1} \neq G$ .

(1) Every non-trivial quotient  $G/N$  of  $G$  is a  $D_\pi^S$ -group (See (1) in the proof of Theorem 4.3).

(2) If  $N$  is a normal  $E_\pi^N$ -subgroup of  $G$ , then  $N = 1$ . In particular,  $X = 1$ .

Suppose that  $N \neq 1$ . Then  $G/N$  of  $G$  is a  $D_\pi^S$ -group by (1) and the choice of  $G$ . Hence  $G$  is a  $D_\pi^S$ -group by Lemma 5.1, which contradicts the choice of  $G$ . Hence we have (2).

(3)  $T \neq G$  (See (3) in the proof of Theorem 5.2).

(4) The hypothesis holds for  $H_{t-1}$  and  $T$ .

Since

$$H_{t-1} = H_1(H_{t-1} \cap T)$$

and

$$|H_{t-1} : H_{t-1} \cap T| = |G : T|,$$

the hypothesis holds for  $H_{t-1}$  by (2). The second assertion of (4) may be proved as (5) in the proof of Theorem 4.1.

(5)  $G$  is a  $C_\pi^S$ -group.

Since  $|G : H_{t-1}|$  is divisible by at most one prime in  $\pi$  and  $|\pi \cap \pi(G)| > 1$ , there is a prime  $p \in \pi \cap \pi(G)$  such that for a Sylow  $p$ -subgroup  $P$  of  $G$  we have  $P \leq H_{t-1}$ , so in view of (2) and Lemma 2.10 we have  $1 < P \leq (H_{t-1})_G$ . Since  $H_{t-1} \neq G$  and the hypothesis holds for  $H_{t-1}$ ,  $H_{t-1}$  is a  $D_\pi^S$ -group by the choice of  $G$ . Hence  $(H_{t-1})_G$  is a  $C_\pi^S$ -group, so  $G$  is a  $C_\pi^S$ -group by Lemma 2.1 (i) and Theorem 5.2.

(6) For any  $i$ ,  $H_i$  permutes with every  $\pi$ -subgroup of  $G$  (see (3) in the proof of Theorem 4.3).

(7)  $V = H_1U$  is a  $D_\pi^S$ -group for any  $\pi$ -subgroup  $U$  of  $G$ .

By (6),  $V$  is a subgroup of  $G$ . Moreover,  $V$  is a  $C_\pi$ -group by Proposition 3.1. We show that the hypothesis holds for  $V$ . Let  $V_\pi$  be a Hall  $\pi$ -subgroup of  $V$ . Then  $V = H_1V_\pi$  and

$$V_i = V \cap H_i = H_1(V_\pi \cap H_i),$$

for all  $i = 1, \dots, t$ . Let  $W$  be any subgroup of  $V_\pi$ . Then

$$V_iW = (H_1(V_\pi \cap H_i))W = H_1(V_\pi \cap H_iW) = H_1W(V_\pi \cap H_i)W(H_1(V_\pi \cap H_i)) = WV_i.$$

It is clear also that  $|V_{i+1} : V_i|$  is divisible by at most one prime in  $\pi$ , for all  $i = 0, \dots, t$ . Hence the hypothesis holds for  $V$ . Suppose that  $G = V$ . In this case, in view of (5), we may suppose that  $V_\pi$  is a soluble Hall  $\pi$ -subgroup of  $G$ . Let  $L$  be a minimal normal subgroup of  $V_\pi$ . Then

$$L^G = L^{V_\pi H_1} = L^{H_1} \leq LH_1 \cap L^{H_1} = L(L^G \cap H_1)$$

and  $L$  is a  $q$ -group for some  $q \in \pi$ . From (2) it follows that for some prime  $p \in \pi$  with  $q \neq p$  we have  $p \in \pi(L^G \cap H_1)$  and a Sylow  $p$ -subgroup  $P$  of  $L^G \cap H_1$  is also a Sylow subgroup of  $L^G$ . For some Sylow  $p$ -subgroup  $G_p$  of  $G$  we have  $P = L^G \cap G_p$ , so

$$L^G \cap H_1G_p = (L^G \cap H_1)(L^G \cap G_p) = (L^G \cap H_1)P = P(L^G \cap H_1).$$

Hence  $P \leq (L^G \cap H_1)_{L^G}$ . Therefore  $G$  has a non-identity subnormal subgroup  $R = (L^G \cap H_1)_{L^G}$  of order divisible by at most one prime in  $\pi$ , which contradicts (2). Hence  $V \neq G$ , so  $V$  is a  $D_\pi^S$ -group by the choice of  $G$ .

*Final contradiction.* Let  $U$  be any  $\pi$ -subgroup of  $G$  and  $V = H_1U$ . Then  $V$  is a  $D_\pi^S$ -subgroup of  $G$  by (7), and  $|G : T| = |V : V \cap T|$  is a  $\pi'$ -number. By (3) and (4),  $T$  is a  $D_\pi$ -group. Hence for some Hall  $\pi$ -subgroup  $V_\pi$  of  $V$  we have  $U \leq V_\pi$ , and  $V_\pi \leq T_\pi$ . But since  $|G : T|$  is a  $\pi'$ -number,  $T_\pi$  is a Hall  $\pi$ -subgroup of  $G$ . Therefore  $G$  is a  $D_\pi^S$ -group.

**Corollary 5.4.** *Suppose that  $G$  has a subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

where  $|H_{i+1} : H_i|$  is divisible by at most one prime in  $\pi$ , for all  $i = 1, \dots, t$ . Suppose also that  $G$  has a subgroup  $T$  such that  $H_1T = G$  and  $|G : T|$  is a  $\pi'$ -number. If  $H_i$  is quasinormal in  $T$ , for all  $i = 1, \dots, t$ , then  $G$  is a  $D_\pi^S$ -group.

**Corollary 5.6** (see S. A. Čuniĥin [21] or [6, Chapter IV, Theorem 18.13]). *If  $G$  has a chief subgroup series*

$$1 = H_0 < H_1 \leq \dots \leq H_{t-1} \leq H_t = G,$$

*where  $|H_{i+1} : H_i|$  is divisible by at most one prime in  $\pi$ , for all  $i = 1, \dots, t$ , then  $G$  is a  $D_\pi^S$ -group.*

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